Braid invariants for non-linear differential equations

Munao, S.

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1 Introduction

In this thesis we investigate topological properties and invariants of a special class of non-linear partial differential equations (PDEs) with applications to the theory of relative braid classes. The three main results can be summarized as follows:

- the Poincaré-Hopf Theorem for relative braid classes;
- the construction of an isomorphism between the braid Floer homology and the braid Morse homology;
- a generalization of the Poincaré-Bendixson Theorem for non-linear Cauchy-Riemann equations.

In the next sections we explain the interplay between braids and differential equations, and why we can exploit the topological properties therein contained. In order to put this into a more general context, we start off with some examples which involve the solution of analytical problems via topological tools.

1.1 Examples of topological invariants in analysis

In this thesis we use topology as a useful tool which can give information on the structure of dynamical systems. Perhaps the first user of topology in differential equations was Poincaré, who developed many of his topological methods while studying ordinary differential equations which arose from certain astronomy problems. His study of autonomous systems

$$\dot{x} = F(x), \qquad x \in \mathbb{R}^2, \quad F \in C^1(\mathbb{R}^2; \mathbb{R}^2).$$

involved looking at the totality of all solutions rather than at particular trajectories as had been the case earlier. This is the context of the famous Poincaré-Bendixson Theorem.

The use of topological techniques in analysis is full of insightful examples. We provide three.

(i) The classical BROUWER DEGREE theory provides a tool that contains information about the zeroes of a continuous function. Its infinite-dimensional generalization, the LERAY-SCHAUDER DEGREE, applies to a special class of operators.

(ii) The POINCARÉ-HOPF FORMULA relates a purely topological concept, i.e. the Euler characteristic of a smooth manifold M, to the index of a vector field on M, which is a purely analytical concept.

(iii) MORSE THEORY also can be put in this framework: one of the consequences of the Morse inequalities is that the number of critical points of any Morse function on a smooth manifold is closely related to the homology of the underlying manifold.

1.1.1 The Brouwer degree and the Leray-Schauder degree

The analytical construction of the (localized) Brouwer degree $\deg(f, \Omega, p)$ of a smooth mapping $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, with Ω open bounded and a regular value $p \notin f(\partial\Omega)$, is defined as

$$\deg(f, \Omega, p) := \sum_{x \in f^{-1}(p)} \operatorname{sgn} J_f(x).$$

Here J_f denotes the Jacobian of f. By approximation one can extend the definition to continuous functions and also to non-regular values.

For such maps the degree being non-vanishing implies the existence of an $x \in \Omega$ such that $f(x) = p$. More importantly the Brouwer degree is invariant under homotopies of functions and of domains. On the base of these properties one can show that degree theory has important implications, among which Brouwer's fixed point Theorem. In full generality the latter states that any Hausdorff topological space homeomorphic to the unit closed ball $B_1(0) \subset \mathbb{R}^n$ has the fixed point property [1].

A straightforward generalization of Brouwer's fixed point Theorem to infinite dimensions, i.e., using the unit ball of an arbitrary Banach space instead of Euclidean space, is not true. The main problem here is that the unit balls in infinite-dimensional Banach spaces are not compact. Nevertheless an infinite dimensional degree theory exists and has been developed by Leray and Schauder. They identified an important class of non-linear operators in a Banach space, the compact perturbations of the identity, for which the problem of contractibility of the sphere could be solved. This extension has been successfully applied to non-linear elliptic boundary value problems, see [42].

1.1.2 The Hairy Ball Theorem and the Poincaré-Hopf formula

The Hairy Ball Theorem states that there is no non-vanishing continuous tangent vector field on even dimensional n-spheres. For ordinary spheres, or 2-spheres, the latter can be rephrased as follows: whenever one attempts to comb a hairy ball flat, there will always be at least one tuft of hair at one point on the ball. The theorem was first stated by Poincaré in the late 19th century and proved in

[1] A Hausdorff topological space X has the fixed point property if every continuous mapping $g : X \to X$ has at least one fixed point.

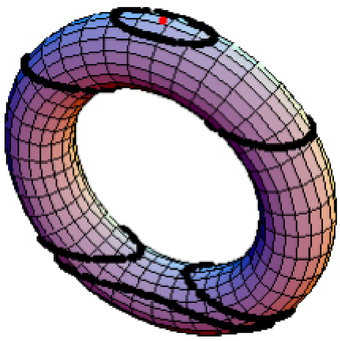

Figure 1.1: The standard two-torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$. In black the sublevel sets of the height function.

1912 by Brouwer. This is famously stated as *you can't comb a hairy ball flat without creating a cowlick,* or sometimes *you can't comb the hair on a coconut*.

From a more advanced point of view, every zero of a vector field has an *index* [2], and it can be shown that for an even dimensional sphere the sum of all of the indices at all of the zeros must be two. Therefore there must be at least one zero. This is a consequence of the Poincaré-Hopf formula. The latter has the form

$$\sum_i \mathrm{ind}_X(x_i) = \chi(M), \tag{1.1}$$

where M is a manifold, X a vector field on M, the sum of the indices is over all the isolated zeroes of X, and $\chi(M)$ is the Euler characteristic of M. One important consequence of (1.1) is that the index of a vector field does not depend on the choice of the vector field, but only on the topology of the manifold M. In the case of the torus, the Euler characteristic is 0; and it is possible to *comb a hairy doughnut flat*. In this regard, it follows that for any compact regular 2-dimensional manifold with non-zero Euler characteristic, any continuous tangent vector field has at least one zero.

1.1.3 Classical Morse Theory

The power of Morse theory is that it provides an analytical framework in which to study the topology of manifolds. One of the classical references is Milnor [38].

Consider the standard embedding of the two-torus $\mathbb{T}^2$ in $\mathbb{R}^3$, as shown in Figure 1.1 and the height function $h : \mathbb{T}^2 \to \mathbb{R}$ which returns the third coordinate of such embedding. This function has four critical points. By studying the sublevel

[2] defined, for an isolated zero, in terms of the mapping degree introduced in the previous section

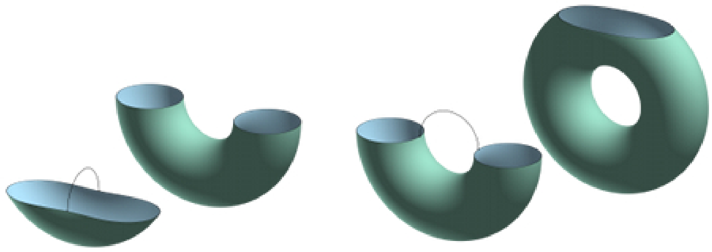

Figure 1.2: Handle decomposition of the sublevel sets of the embedded two-torus.

sets $M_c = h^{-1}((-\infty, c))$ we realize that the topology of M_c does not change as long as c does not pass a critical value of f. When c crosses a critical value, the topology changes. Morse theory is the study of this phenomenon. More generally, if a manifold has a non-trivial homotopy type, the sublevel set M_∞ has a non-trivial homotopy type and therefore f must have critical points. The Morse inequalities [9] are a concise formulation of this, relating the minimum number of critical points of a function to the homology of the underlying manifold. They imply furthermore (1.1). One can prove that the homotopy type of a sublevel set changes exactly by attaching an n-cell (or an n-handle as in Figure 1.2) where n is given by the nature of the critical point, i.e., depending whether the critical point is a minimum, maximum, or a saddle point. One builds up a CW-complex in this manner, which captures the homotopy type of the manifold. For this to work the function f needs to satisfy certain properties, which are contained in the concept of a Morse function.

1.2 Braids and braid diagrams

In this section we begin with informal definitions of braids, braid classes and relative braid classes in three different contexts. All three settings are closely related and we will point out their relations. We will mainly follow [53].

1.2.1 Braids on $\mathbb{D}^2$

Consider the standard 2-disc $\mathbb{D}^2$ (with coordinates $x = (p, q) \in \mathbb{D}^2$) in the plane and the cylinder $C = [0,1] \times \mathbb{D}^2$. An unordered collection of continuous functions $x = \{x^1(t), \ldots, x^m(t)\}, x^k : [0,1] \to \mathbb{D}^2$ (called strands) is called a braid on the 2-disc $\mathbb{D}^2$ if:

(i) $x^k(t+1) = x^{\sigma(k)}(t)$ for some permutation $\sigma \in S_m$, and

(ii) $x^k(t) \neq x^h(t)$ for all $k \neq h$ and all $t \in [0,1]$.

The set of all braids on $\mathbb{D}^2$ homotopic to x is denoted by $[x]_{\mathbb{D}^2}$ and is called a braid class. We will often use, for such braids, the terminology *bounded* braids, since for all $x_0 \in [x]_{\mathbb{D}^2}$ we have $|x_0| \leq 1$. A way to visualize a braid is to consider a so-called braid diagram in the plane. The latter is obtained by projecting the cylinder C onto a plane of the form $[0,1] \times L$, where is $L \subset \mathbb{R}$ is a diameter of $\mathbb{D}^2$. If we denote the projection by $\pi : \mathbb{D}^2 \to L$, then two strands $x^k(t)$ and $x^h(t)$ have a positive crossing in the projection at $\pi x^k(t_0) = \pi x^h(t_0)$ if $x^k - x^h$ rotates counter clockwise about the origin, for small interval of times t around t_0. A negative crossing corresponds to a clockwise rotation. Now consider special collections of the form $\{x(t), y^1(t), \ldots, y^m(t)\}$, with $x = \{x(t)\}$ a periodic function on $[0,1]$, with values in $\mathbb{D}^2$ and $y = \{y^1(t), \ldots, y^m(t)\}$ as above. Denote such collections by $x \operatorname{rel} y$ and assume that they are braids with $1+m$ strands. Since we singled out two braid components we denote the braid class containing $x \operatorname{rel} y$ by $[x \operatorname{rel} y]_{\mathbb{D}^2}$. The latter is called a relative braid class, abbreviated RBC. The component y is called the *skeleton* of the relative braid class. We refer to the x-component as the *free part*. If we take the skeleton y to be fixed, then the set of periodic functions x for which $x \operatorname{rel} y$ is a braid is denoted by $[x]_{\mathbb{D}^2} \operatorname{rel} y$ and is called a relative braid class fiber. On $[x]_{\mathbb{D}^2} \operatorname{rel} y$ we consider the C^0 topology. The space $[x \operatorname{rel} y]_{\mathbb{D}^2}$ is a fibered space over $[y]_{\mathbb{D}^2}$ and the relative braid class $[x]_{\mathbb{D}^2} \operatorname{rel} y$ is a fiber in $[x \operatorname{rel} y]_{\mathbb{D}^2}$. The intertwining between x and y gives rise to different braid classes. A relative braid class is called PROPER if x can not be deformed, or 'collapsed', onto any y components, nor onto the boundary $\partial\mathbb{D}^2$. We abbreviate proper relative relative braid classes as PRBCes. In this thesis we consider only relative braids, whose free part is composed by only *one* strand, but we can easily generalize the notion of relative braid (classes) with x consisting of n strands.

1.2.2 Braid diagrams in dimension 1

In the special case that strands $x(t)$ are of the form $x_L(t) = (q_t(t), q(t))$ the projection onto the q-coordinate provides a representation of a braid in terms of graphs. Such strands satisfy the property that they lie in the kernel of the one-form

$$\alpha = dq - pdt,$$

which is known as the Legendrian property. An unordered collection of functions $Q = \{Q^1(t), \ldots, Q^m(t)\}, Q^j : [0,1] \to [-1,1], j = 1, \ldots, m$ is called a (bounded) braid diagram or, equivalently a (bounded) LEGENDRIAN braid if

(i) $Q^k(t+1) = Q^{\sigma(k)}(t)$ for some $\sigma \in S_m$, and

(ii) all graphs $Q^k(t)$ intersect transversally.

The set of all braid diagrams isotopic to Q is denoted by $[Q]_{[-1,1]}$. As before we also consider collections of the form $q \operatorname{rel} Q = \{q(t), Q^1(t), \ldots, Q^m(t)\}$ and the

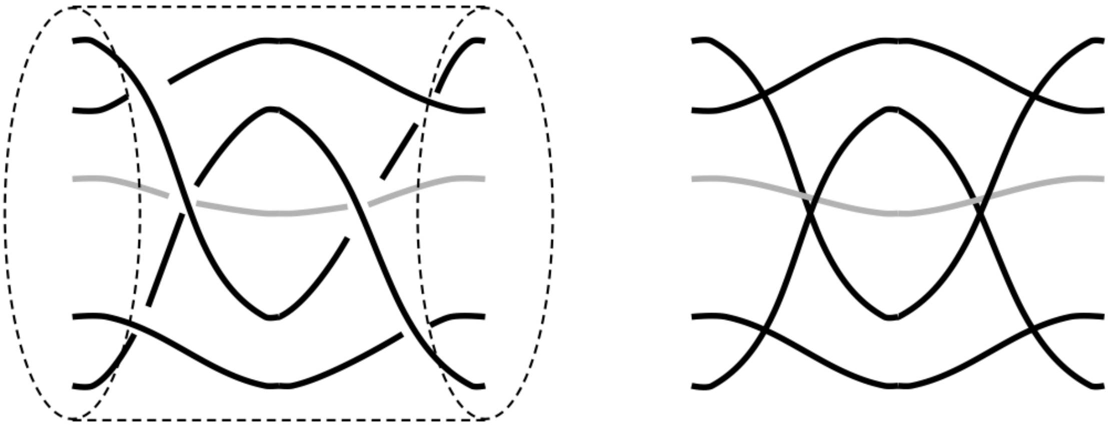

Figure 1.3: A positive relative braid and its Legendrian projection.

associated (bounded) relative braid classes $[q \operatorname{rel} Q]_{[-1,1]}$ and $[q]_{[-1,1]} \operatorname{rel} Q$ (fibers). In order to slim the notation for bounded Legendrian RBC we will write simply $[q \operatorname{rel} Q]$ and for fibers $[q] \operatorname{rel} Q$, instead of $[q \operatorname{rel} Q]_{[-1,1]}$ and $[q]_{[-1,1]} \operatorname{rel} Q$ respectively. It is immediate that these Legendrian braid classes are a subset of the braid classes on $\mathbb{D}^2$. The Legendrian constraints implies that all crossings of strands are positive.

As in the case of $\mathbb{D}^2$, the intertwining between q and Q yields different braid classes. In this case the notion of *proper* translates into the following condition. We say that a relative Legendrian braid class is PROPER if the strand q cannot be deformed onto any of the strands Q^k, for all $k = 1, \dots, m$ nor onto the constant strands ± 1.

1.2.3 Discrete braid diagrams

Yet another simplification is obtained by considering piecewise linear functions connecting the points $q_i = q(i/d), i = 0, \dots, d$. We represent such piecewise linear functions by sequences $q_D = \{q_i\}_{i=0,\dots,d}$. Both the sequences and their linear piecewise extension will be denoted by the same symbol q_D. An unordered collection of sequences $Q_D = \{Q_D^1, \dots, Q_D^m\} = \{\{Q_i^1\}, \dots, \{Q_i^m\}\}_{i=0,\dots,d}$ is called a discrete, or PIECEWISE LINEAR BRAID DIAGRAM if

(i) $Q_{i+1}^k = Q_i^{\sigma(k)}$, for some permutation $\sigma \in S_m$, and for all $i = 0, \dots, d$

(ii) all the graphs $Q^k(t)$ intersect transversally [3].

[3] in this setting we say that an intersection is transverse if $(Q_{i-1}^k - Q_{i-1}^{k'})(Q_{i+1}^k - Q_{i+1}^{k'}) > 0$ whenever $Q_i^k = Q_i^{k'}$.

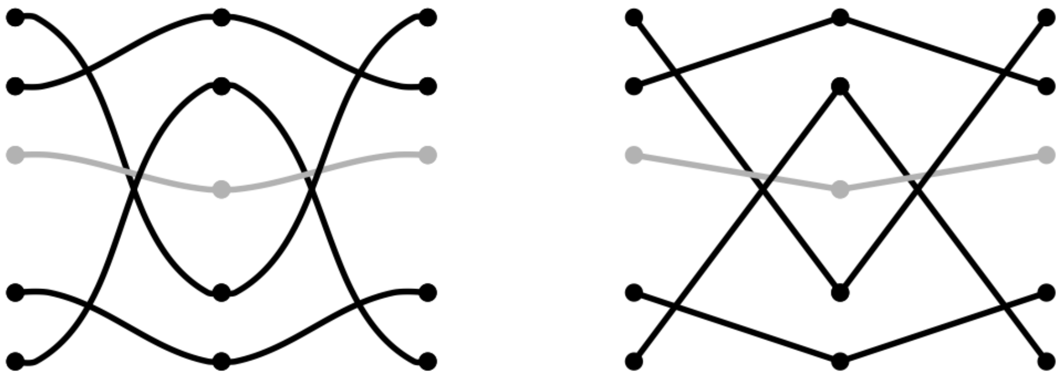

Figure 1.4: A Legendrian relative braid and its discretization.

The set of the equivalents classes, via isotopy, fixing the endpoints is denoted by $[Q_D]$. Crossing in this setting are also marked as positive. Collections of the form $q_D \,\mathrm{rel}\, Q_D = \{q_D, Q_D^1, \ldots Q_D^m\}$ and the associated relative braid class are denoted by $[q_D \,\mathrm{rel}\, Q_D]$ are the fibers by $[q_D] \,\mathrm{rel}\, Q_D$. As before, we say that a class of discrete braid diagrams is PROPER if the piecewise linear strand q_D cannot be deformed onto any of the strands Q_D^k, for all $k = 1, \ldots, m$ and the strand q_D cannot de deformed onto the constant sequence ± 1.

1.2.1. Remark. Properness is a topological condition that descents from braids on $\mathbb{D}^2$ to discrete braids, i.e. properness of $[x \,\mathrm{rel}\, y]_{\mathbb{D}^2}$ implies

$$[x \,\mathrm{rel}\, y]_{\mathbb{D}^2} \implies [q \,\mathrm{rel}\, Q] \implies [q_D \,\mathrm{rel}\, Q_D].$$

The implications do not necessarily go in the opposite direction.

1.3 State of the art: braids and PDEs

The use of braids in dynamics is not without precedent (see e.g. [27], [28], [29], [39], [50]), in particular if applied to the theory of topological forcing in dimension two and three ([26], [50], [51]). How do braids evolve and under which equations this motion is ruled? We explain this in the next three paragraphs. An important motivation for using braid theory in dynamics comes from the comparison principle, which essentially states that if we evolve a braid in time, the complexity of the braid diminishes. The comparison principle motivates the choice of the Cauchy-Riemann equation for braids in $\mathbb{D}^2$, the choice of the heat flow for Legendrian braids in dimension 1, and the choice of discrete parabolic relations in the discrete case.

1.3.1 The Cauchy-Riemann equations

The non-linear equations

$$u_s - J(s,t)(u_t - X_H(t,u)), \quad u : \mathbb{R} \times S^1 \to \mathbb{D}^2 \tag{1.2}$$

are called the Cauchy-Riemann equations, or, abbreviated, non-linear CRE. The parameters J and H are called almost complex structure and Hamiltonian respectively. An almost complex structure is a smooth map $J : \mathbb{R} \times S^1 \to \mathrm{Sp}(2,\mathbb{R})$ such that $J(s,t)^2 = -\mathrm{Id}$, for all $(s,t) \in \mathbb{R} \times S^1$ (here $\mathrm{Sp}(2,\mathbb{R})$ denotes the symplectic group of degree 2 over $\mathbb{R}$). We consider the class of constant almost complex structures and we denote it by $\mathscr{J}$. Regarding the Hamiltonian function $H : S^1 \times \mathbb{D}^2 \to \mathbb{R}$, we assume that $H(t,x) = 0$ for all $x \in \partial\mathbb{D}^2$ and all $t \in \mathbb{R}$ and we call this class of Hamiltonians $\mathscr{H}$. The Hamiltonian function H gives rise to the *Hamiltonian* vector field X_H.

For a braid x the total crossing number $\mathrm{Cross}(x)$ is defined as the number of positive minus the number of negative crossings, i.e.

$$\mathrm{Cross}(x) := \#\{\text{positive crossings}\} - \#\{\text{negative crossings}\}.$$

For relative braids this number is denoted by $\mathrm{Cross}(x \,\mathrm{rel}\, y)$ and it is an invariant of the relative braid class $[x \,\mathrm{rel}\, y]_{\mathbb{D}^2}$. Let $[x]_{\mathbb{D}^2} \,\mathrm{rel}\, y$ be a relative braid class fiber with skeleton y, then we can choose Hamiltonians H, such that the skeletal strands are solutions of the s-stationary equations $y_t = X_H(t,y)$. Let $u(s,\cdot) \,\mathrm{rel}\, y$ denote a local solution in s of the Cauchy-Riemann equations, then

$$\mathrm{Cross}(u(s_1,\cdot) \,\mathrm{rel}\, y)| \leq \mathrm{Cross}(u(s_0,\cdot) \,\mathrm{rel}\, y) \quad \text{for all } s_1 \geq s_0.$$

This is also known in literature as the Monotonicity Lemma (see [50]): in essence along solutions $u(s,t)$ of the non-linear CRE (1.2), the number $\mathrm{Cross}(u(s,) \,\mathrm{rel}\, y)$ is non-increasing. In other words, along flow-lines of the non-linear CRE positive crossings can evolve into negative crossings, but not vice-versa. If we consider braid classes which are proper they yield isolating sets for the dynamics: a bounded solution $u(s,\cdot) \in [x]_{\mathbb{D}^2} \,\mathrm{rel}\, y$ with $[x]_{\mathbb{D}^2} \,\mathrm{rel}\, y$ a proper fiber stays away both from any of the y components and from $\partial\mathbb{D}^2$.

1.3.2 The heat flow

Consider the scalar parabolic equation, or the non-linear heat flow equation

$$v_s - v_{tt} + v - \partial_v W(t,v) = 0, \quad v : \mathbb{R} \times S^1 \to [-1,1]. \tag{1.3}$$

For the non-linearity W we assume the following hypotheses: $W \in C^\infty(S^1 \times [-1,1];\mathbb{R})$ and $\partial_v W(t,\pm 1) = \pm 1$ for all $t \in S^1$. Equation (1.3), unlike (1.2), generates a local semi-flow ψ^s on the space of periodic function $C^0(S^1;[-1,1])$.

Let Q be a braid diagram of dimension 1 on m strands, we can define the analogue of the crossing number for x as the intersection number $\mathrm{I}(Q)$ as it follows:

$$\mathrm{I}(Q) := \#\{\text{total number of crossings}\}.$$

Since, by the Legendrian constraint, all intersections correspond to positive crossings, the total intersection number is equal to the crossing number defined above. This means that if we define $y = (Q_t, Q)$ then

$$\mathrm{Cross}(y) = \mathrm{I}(Q).$$

The classical lap-number property [6] of non-linear scalar heat equations states that the number of intersections between two graphs can only decrease in time s, as s increases.

We now apply this principle to Legendrian braid classes. Let $[q]\,\mathrm{rel}\,Q$ a Legendrian RBC fiber with skeleton Q and suppose that we can choose the non-linearity U such that the skeletal strands are solutions of the stationary equation $Q_{tt} - Q + \partial_Q W(t,Q) = 0$. Denote by $v(s,\cdot)\,\mathrm{rel}\,Q$ local solutions of the heat equation, then, as in the elliptic case, then

$$\mathrm{I}(v(s_1,\cdot)\,\mathrm{rel}\,Q) \le \mathrm{I}(v(s_0,\cdot)\,\mathrm{rel}\,Q) \quad \text{for all } s_1 \ge s_0$$

If we consider Legendrian braid classes that are *proper* they yield isolating sets for the dynamics: also in this case a bounded solution $v(s,\cdot) \in [q]\,\mathrm{rel}\,Q$ with $[q]\,\mathrm{rel}\,Q$ a proper Legendrian fiber stays away both from each of the Q components and from the constant strands ± 1 (this property is also called isolation of proper braid classes).

1.3.3 Discrete parabolic relations

In the discrete setting the dynamics that respect the braids consists of the discrete parabolic equations. These are recurrence relations on the space of discretized braid diagram and consist of nearest neighbor interaction. They resemble spacial discretizations of parabolic equations. For a k-strand braid diagram on d points, the discrete parabolic relations are given by

$$\frac{d}{ds} v_i^\alpha = \mathcal{R}_i(v_{i-1}^\alpha, v_i^\alpha, v_{i+1}^\alpha), \quad \text{for all } i = 0, \dots, d-1, \tag{1.4}$$

for every $\alpha = 1, \ldots, k$. On $\mathcal{R}_i$ we assume the following: $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i > 0$; $\mathcal{R}_{i+d} = \mathcal{R}_i$, for all i.

If we restrict the range of the sequences $v_D^\alpha, \alpha = 1, \ldots k$ to the interval $[-1, 1]$, then Equation (1.4) generates a flow ϕ^s on the space $\mathcal{D}_d^k$ of k-tuples of d-periodic sequences. This flow will be referred as parabolic flow on $\mathcal{D}_d^k$. If we furthermore assume that $\mathcal{R}_i(-1,-1,-1) = \mathcal{R}_i(1,1,1) = 0$ for all i, the constant sequences ± 1 are stationary for the flow ϕ^s.

There is a discrete analogue of the crossing number and the intersection number. Recall that any discrete braid diagram (of k-strands) can be expressed in terms of the (positive) generators $\{\sigma_j\}_{j=1}^{k-1}$ of the braid group $\mathbf{B}_k$. While this word is not necessarily unique, the length of the word is, as one can easily see from the representation of $\mathbf{B}_k$. As in the previous cases, we consider piecewise linear braids that are composed by a free part and a skeletal part, and we denote them by $q_D \operatorname{rel} Q_D$. Note that the skeletal part may consist of multiple (say m) piecewise linear strands, i.e. $Q_D = \{Q_D^1, \ldots, Q_D^m\}$, while we consider the free strands to be only of 1 strand. The length of a closed braid in the generators σ_j is thus precisely the word metric $\ell(Q_D)$ from geometric group theory. The geometric interpretation of $\ell(Q_D)$ for a piecewise linear braid Q_D is the number of pairwise strand crossings in the diagram Q_D. This means that if we discretize a positive braid diagram Q in dimension 1 and we call the discretization Q_D then

$$\ell(Q_D) = \mathrm{I}(Q).$$

A result in [28] shows that, as for the continuous case, the word length can only decrease as time s increases.

We now apply this principle to discrete braid diagram. Let $[q_D] \operatorname{rel} Q_D$ a discretized RBC fiber with skeleton Q_D and suppose that we can choose $\mathcal{R}_i$ such that the skeletal strands are solutions of the stationary equation for all $\alpha = 1, \ldots, m$ $\mathcal{R}_i(Q_{i-1}^\alpha, Q_i^\alpha, Q_{i+1}^\alpha) = 0$, for $i = 0, \ldots, d-1$ (and by periodicity $Q_0^\alpha = Q_d^\alpha$ for all $\alpha = 1, \ldots, m$). Denote by $v_D(s) \operatorname{rel} Q$ local solutions of (1.4), then, as in the elliptic case, and in the continuous parabolic case we have

$$\ell(v_D(s_1) \operatorname{rel} Q) \leq \ell(v_D(s_0) \operatorname{rel} Q) \quad \text{for all } s_1 \geq s_0$$

This was shown in [28]. Also in this case, if we consider discrete braid classes which are proper they yield isolating sets for the dynamics.

1.4 Braid invariants

In the previous section we linked the three types of braid classes to natural dynamical systems associated with these braid classes. They all share the properties that proper braid classes yield isolating sets for the dynamics.

Floer's approach, used in the beginning to solve the Arnol'd conjecture, develops a Morse type theory for the Hamiltonian action

$$\mathscr{A}_H(x) = \int_0^1 \tfrac{1}{2}\langle Jx, x_t\rangle\, dt - \int_0^1 H(t, x(t))\, dt.$$

This applies to the non-linear CRE. The variational structure for the heat flow is given by the action

$$\mathscr{L}_U(q) = \int_0^1 \tfrac{1}{2}|q_t|^2 + \tfrac{1}{2}|q|^2\, dt - \int_0^1 U(t, q(t))\, dt$$

and takes the name of Lagrangian action functional. A discrete variational principle for discrete parabolic equations is given by the action

$$\mathscr{W}(\{q_i\}) = \sum_{i=0}^{d-1} S_i(q_i, q_{i+1}),$$

where S_i are smooth functions on $[-1,1]\times[-1,1]$ with the property that $\partial_1\partial_2 S_i > 0$. In this case $\mathcal{R}_i = \partial_2 S_{i-1} + \partial_1 S_i$. All the equations introduced above are now gradient flow equations and we carry out Floer's procedure.

1.4.1 Floer homology, Morse homology, Conley homology for proper relative braid classes

Let us explain the basic ingredients of Floer theory for the Cauchy-Riemann equations. The same applies to the other two cases. We should emphasize that the ingredients for obtaining respectively Floer homology, Morse homology and Conley homology are the same, but working out the details is very delicate and sometimes very tedious. Denote the set of bounded solutions of Equation (1.2) in a braid class fiber $[x]_{\mathbb{D}^2}\,\mathrm{rel}\,y$, that exists for all $s \in \mathbb{R}$, by $\mathscr{M}([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y; J, H)$. The image under the mapping $u \mapsto u(0,\cdot)$ is denoted by $\mathscr{S}([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y; J, H) \subset C(S^1; \mathbb{R}^2)$.

- *Compactness.* Consider a PRBC and the set $\mathscr{M}$ of bounded solutions of the Cauchy-Riemann equations in the considered braid class. Elliptic regularity guarantees that the spaces $\mathscr{M}$ and $\mathscr{S}$ are compact with respect the appropriate topologies and properness insures that $\mathscr{S}$ is isolated. Compactness and isolation hold in all the three cases.

- *Genericity of critical points.* For a generic choice of Hamiltonians H in the class $\mathscr{H}$ (where $\mathscr{H}$ has been introduced in Section 1.3.1) for which the skeletal strands y are solutions of the associated Hamilton equations, the critical points of $\mathscr{A}_H$ in the proper relative braid class $[x]_{\mathbb{D}^2} \operatorname{rel} y$ are non-degenerate. Hence the set of critical points in $[x]_{\mathbb{D}^2} \operatorname{rel} y$, which we denote by $\operatorname{Crit}_H([x]_{\mathbb{D}^2} \operatorname{rel} y)$, consists only of finitely many isolated points. Notice that no non-degeneracy condition is imposed on the y strands. The fact that there are only finitely many isolated critical points in a class holds also for the other cases.

- *Genericity of connecting orbits.* The gradient structure of the Cauchy-Riemann equations implies that $\mathscr{M}$ is the union of the space of connecting orbit:

$$\mathscr{M}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) = \bigcup_{x^\pm \in \operatorname{Crit}_H([x]_{\mathbb{D}^2} \operatorname{rel} y)} \mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H),$$

where $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H)$ is the subspace of bounded solutions of Equation (1.2) with limits x^- and x^+ for $s \to \pm\infty$. It can be proven that for generic choice of J and H, the space of connecting orbit are smooth finite dimensional manifolds without boundary.

- *Index function.* One can establish a grading $\mu(x)$ on the non-degenerate elements of $\operatorname{Crit}_H([x]_{\mathbb{D}^2} \operatorname{rel} y)$ in such a way the the dimension of $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H)$ is given by the formula

$$\dim \mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) = \mu(x^-) - \mu(x^+).$$

This theory is based on the theory of Fredholm operators and holds in all cases. For the Cauchy-Riemann equations we chose μ to be the Conley-Zender index, for the heat flow the classical Morse index and the same for the case of discrete parabolic equations.

- *Chain complex and its homology.* The construction of the chain complex and therefore the Floer homology has become a standard procedure ([23]). By the compactness and genericity $\operatorname{Crit}_H([x]_{\mathbb{D}^2} \operatorname{rel} y)$ is finite and we define the chain groups $C_k([x]_{\mathbb{D}^2} \operatorname{rel} y)$ as formal sum $\sum_j \alpha_j x_j$ with coefficients $\alpha_j \in \mathbb{Z}_2$. A boundary operator is defined by the formula

$$\partial_k x = \sum_{\mu(x')=k-1} n(x, x')x',$$

where $n(x,x')$ is the number of elements (modulo 2) in $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H)$ with $\mu(x^-) - \mu(x^+) = 1$. Genericity and compactness imply that this number is finite. Proving that ∂_k is a boundary operator is equivalent to showing that

$$\partial_{k-1} \circ \partial_k = 0.$$

The composition counts the number of broken trajectories, i.e. the number of elements in the set

$$\bigcup_{\mu(x')=k-1} \left(\mathscr{M}^{x^-,x'}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) \times \mathscr{M}^{x',x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) \right).$$

The space $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H)/\mathbb{R}$, with $\mu(x^-) - \mu(x^+) = 2$, is a manifold without boundary of dimension 1 and the Floer's gluing construction reveals that if $\mathscr{M}^{x^-,x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H)/\mathbb{R}$ is not compact then the manifold can be compactified to manifold with boundary diffeomorphic to $[0,1]$ by adding broken trajectories in

$$\bigcup_{\mu(x')=k-1} \left(\mathscr{M}^{x^-,x'}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) \times \mathscr{M}^{x',x^+}([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) \right).$$

The gluing construction also reveals that the procedure is surjective and thus the number of broken trajectories is even, thus $\partial_{k-1} \circ \partial_k = 0$. In the end this proves that (C_*, ∂_*) is a chain complex and its homology is well-defined and finite.

We define

$$\mathrm{HF}_k([x]_{\mathbb{D}^2} \operatorname{rel} y; J, H) := H_k(C_*, \partial_*).$$

Different choices of $H \in \mathscr{H}$ and of $J \in \mathscr{J}$ ($\mathscr{H}$ and $\mathscr{J}$ have been defined in Section 1.3.1) yield isomorphic Floer homologies and

$$\mathrm{HF}_*([x]_{\mathbb{D}^2} \operatorname{rel} y) = \varprojlim \mathrm{HF}_*([x]_{\mathbb{D}^2} \operatorname{rel} y; H, J),$$

where the inverse limit is defined with respect to the canonical isomorphisms $a_k(H,H') : \mathrm{HF}_k([x] \operatorname{rel} y; H, J) \to \mathrm{HF}_k([x] \operatorname{rel} y; H', J)$ and $b_k(J,J') : \mathrm{HF}_k([x] \operatorname{rel} y; H, J) \to \mathrm{HF}_k([x] \operatorname{rel} y; H, J')$. Some properties are (see [50] for the proofs):

(i) the groups $\mathrm{HF}_k([x]_{\mathbb{D}^2} \operatorname{rel} y)$ are defined for all $k \in \mathbb{Z}$ and are finite, i.e. $\mathbb{Z}_2^d$ for some $d \geq 0$;

(ii) the groups $\mathrm{HF}_k([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y)$ are invariants for the fibers in the same relative braid class $[x\,\mathrm{rel}\,y]_{\mathbb{D}^2}$, i.e. if $x\,\mathrm{rel}\,y \sim x'\,\mathrm{rel}\,y'$, then $\mathrm{HF}_k([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y) \cong \mathrm{HF}_k([x']_{\mathbb{D}^2}\,\mathrm{rel}\,y')$. For this reason we will write $\mathrm{HF}_*([x\,\mathrm{rel}\,y]_{\mathbb{D}^2})$;

(iii) if $(x\,\mathrm{rel}\,y)\cdot\Delta^{2\ell}$ denotes composition with ℓ full twists, then $\mathrm{HF}_k([(x\,\mathrm{rel}\,y)\cdot\Delta^{2\ell}]_{\mathbb{D}^2}) \cong \mathrm{HF}_{k-2\ell}([x\,\mathrm{rel}\,y]_{\mathbb{D}^2})$.

A similar construction can be carried out for the heat flow equation and the discrete parabolic equation leading to Morse and Conley homology, respectively

$$\mathrm{HM}_*([q\,\mathrm{rel}\,Q]) \quad \text{and} \quad \mathrm{HC}_*([q_D\,\mathrm{rel}\,Q_D]).$$

The latter is isomorphic to the homological Conley index. The former will be referred to as the Morse homology of $[q\,\mathrm{rel}\,Q]$ and the latter as the homological Conley index of $[q_D\,\mathrm{rel}\,Q_D]$. Note that properties (i) and (ii) continue to hold in the three different settings.

1.5 Discussion of the results

Since the construction of these three topological invariants is so similar in the three cases, the first question that arises is whether these three topological invariants are related. We give a (partial) answer to this question in this thesis. In the following subsections we analyze the main results contained in this manuscript. The first two results go towards the direction of linking topological invariants for discrete braids to those concerning continuous ones. We go even beyond this aim: in Chapter 2 we link the Euler-Floer characteristic to a non-variational problem, a novelty in the panorama of the Floer context. The last result, i.e. the Poincaré-Bendixson Theorem for non-linear Cauchy-Riemann equations, is more a topological property characterizing these equations.

1.5.1 The Euler-Floer characteristic and periodic point of two-dimensional diffeomorphisms

Endow the 2-disc $\mathbb{D}^2$ with the standard symplectic form $\omega = dp\wedge dq$ and choose a Hamiltonian function H in the class $\mathscr{H}$. Define the time-dependent Hamiltonian vector field X_H via the relation

$$dH = \omega(X_H,\cdot).$$

Solving the initial value problem associated to the vector field X_H, i.e.

$$\begin{cases} \dfrac{dx}{dt} = X_H(t,x) \\ x(0) = x_0 \end{cases} \tag{1.5}$$

gives rise to a smooth family of Hamiltonian symplectomorphisms (i.e. diffeomorphisms that preserve the area form ω and originated from X_H) denoted by $\psi_H : \mathbb{R} \times \mathbb{D}^2 \to \mathbb{D}^2$. The time-1 map $f = \psi_H(1,\cdot)$ is orientation preserving and exactly homotopic to the identity according to the nomenclature introduced in [13]. There is a one-to-one correspondence

$$k\text{-periodic points of } f \overset{1-1}{\longleftrightarrow} \text{period-}k \text{ orbits of } \psi_H.$$

The latter holds because $f^k(x) = x$ if and only if $\{\psi_H(t,x), t \in [0,1]\}$ is a closed orbit of period k. The relation between braids and symplectomorphisms is explained as follows. Let $x \in \mathbb{D}^2$ be a k-periodic point, i.e. $f^k(x) = x$, $k \geq 1$, the minimal period. Then the set $A_k = \{x, f(x), \ldots, f^{k-1}(x)\}$ satisfies $f(A_k) = \{f(x), f^2(x), \ldots, f^k(x) = x\} = A_k$, and a periodic point is thus represented by an element $A_k \in \mathbf{C}_k(\mathbb{D}^2)$, the configuration space of k distinct points in $\mathbb{D}^2$. Any invariant set A_k of f of cardinality k is a point in $\mathbf{C}_k(\mathbb{D}^2)$ and gives rise to a k-strand braid via $t \mapsto \psi(t, A_k)$. Summarizing, a k-periodic point $x \in \mathbb{D}^2$ gives rise to an invariant set $A_k := \{f(x), \ldots, f^{k-1}(x), f^k(x) = x\}$ for f, i.e. $f(A_k) = A_k$. On the other hand, if there exists a $k \in \mathbb{N}$ and distinct points $x_1, \ldots, x_k \in \mathbb{D}^2$, such that the set $A_k := \{x_1, \ldots, x_k\}$ is invariant for f, this does not imply necessarily that there exists one k-periodic point, but that there exists a collection of periodic points $x^1_1, \ldots x^1_{k_1}, x^2_1, \ldots x^2_{k_2}, \ldots, x^\ell_1, \ldots, x^\ell_{k_\ell}$ with $\sum_\ell k_\ell = k$ (to visualize this, see Figure 1.5, where there a braid with 3 strands and 2 components is represented: the diffeomorphism does not have one 3-periodic point, but 1 1-periodic and 2 2-periodic points).

In [50] the authors show that, under the hypotheses that f has an invariant set A_m representing the m-strand braid class $[y]_{\mathbb{D}^2}$, for any proper relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$ for which the braid Floer homology $\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2}) \neq 0$, there exists an invariant set A'_n for f such that the union $A_m \cup A'_n$ represents the relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$. The latter is a forcing result: if the braid Floer homology of associated proper relative braid classes is non-trivial, then additional periodic points of the time-1 map of the Hamiltonian family of symplectomorphisms induced by the Hamilton equations are forced to exist. We stress that different braid classes yield different periodic points.

As explained so far, for any given proper relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$ the Floer homology $\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2})$ is well-defined and applicable to Hamiltonian systems and area-preserving maps of the 2-disc. Two immediate questions that

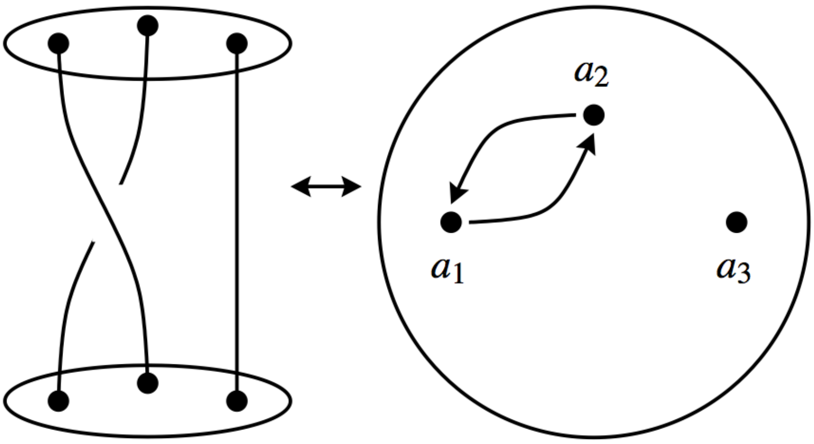

Figure 1.5: Here the braid on the left has two components and three strands. The diffeomorphism is on the right and has one point of period one and two points of period two. The two components of the braid are generated by two points of period two and one of period one. The fact that the braid has three strands does not necessarily imply that the diffeomorphism has one periodic point of period three (source [56]).

come to mind are: *Can the invariant be applied to more general systems and mappings of the 2-disc, and to what extend can the invariants be computed?*

We give a partial answer to this question in Chapter 2 and we summarize our result in this section.

The construction of HF_*, as it is presented in this thesis, fails when X is arbitrary. The main reason is simple: Equation (1.5) relies strongly on a variational principle, one-periodic solutions are critical points of an action functional. By replacing X_H with an arbitrary X the variational structure is lost, and, so far, Floer theory has never been applied in a non-variational setting. The project of building a non-variational Floer theory would certainly be challenging, and there is hope for this to work, also in light of our results presented in Chapter 4. Turning back to our problem, not everything is lost. By substituting in (1.5) a non-Hamiltonian vector field X we obtain

$$\begin{cases} \dfrac{dx}{dt} = X(t,x) \\ x(0) = x_0. \end{cases} \tag{1.6}$$

Under the hypotheses that X is one-periodic ($X(t,x) = X(t+1,x)$) and tangent to the boundary $\partial\mathbb{D}^2$ ($X(t,x)\cdot\nu = 0$ for all $x \in \partial\mathbb{D}^2$, where ν is the outward unit normal on $\partial\mathbb{D}^2$)[4], the system (1.6) gives rise to a smooth family of diffeomorphisms $\phi(t,\cdot) : \mathbb{D}^2 \to \mathbb{D}^2$, whose time 1-map $g = \phi(1,\cdot)$ is orientation preserving. The

[4]In case $X = X_H$ this means that $H \in \mathscr{H}$

one-to-one correspondence between period-m points of g and m-periodic orbits of ϕ^t still holds. Note that g has less structure than f, namely it is *only* a diffeomorphism and *not* a Hamiltonian symplectomorphism in general. By assuming that g has an invariant set B_m that consists of m distinct points in $\mathbb{D}^2$, then B_m gives rise to a m-strand braid, exactly in the same manner as for symplectomorphims.

In Chapter 2 we show that Problems (1.5) and (1.6) can be rephrased into problems "à la Leray-Schauder" in the following way. Multiplying by J, adding $\mu x, \mu \neq 2\pi\mathbb{Z}$ on both sides of (1.5) and (1.6) and inverting $(J\frac{d}{dt} + \mu)$ we obtain respectively

$$\Phi_{\mu,H}(x) = x - (J\tfrac{d}{dt} + \mu)^{-1}(JX_H(x,t) + \mu x) = 0$$

and

$$\Phi_{\mu}(x) = x - (J\tfrac{d}{dt} + \mu)^{-1}(JX(x,t) + \mu x) = 0.$$

In this regard both maps $\Phi_{\mu,H}$ and Φ_μ are in the form "identity minus compact". Now, by assuming y to be the skeleton for X (i.e. $y_t^j = X(t, y^j), j = 1, \dots, m$)and looking at periodic solutions in a *proper* relative braid class fiber $\Omega := [x]_{\mathbb{D}^2} \operatorname{rel} y$, we prove that isolation is preserved also for the non-variational case: in other words solutions that are contained in Ω stay away from the elements of the skeleton y and from the boundary $\partial\mathbb{D}^2$. The Leray-Schauder degree $\deg_{LS}(\Phi_\mu, \Omega, 0)$ is therefore well-defined. By assuming that y is also the skeleton for X_H (such a Hamiltonian function can always be constructed, see [50]) we perform a linear homotopy $X_\alpha = (1-\alpha)X + \alpha X_H,\ \alpha \in [0,1]$. For such homotopy X_α, y it is an admissible skeleton, since it is a skeleton for both X and X_H. Associated with the homotopy X_α we define the homotopy of maps

$$\Phi_{\mu,\alpha}(x) = x - (J\tfrac{d}{dt} + \mu)^{-1}(JX_\alpha(x,t) + \mu x), \quad \alpha \in [0,1].$$

We observe that isolation is preserved *for all* $\alpha \in [0,1]$. By the homotopy invariance of the Leray-Schauder degree we have

$$\deg_{LS}(\Phi_\mu, \Omega, 0) = \deg_{LS}(\Phi_{\mu,\alpha}, \Omega, 0) = \deg_{LS}(\Phi_{\mu,H}, \Omega, 0).$$

By linearizing Φ_μ around a *non-degenerate* solution $x \in \Omega$ and gauging $\Phi'_{\mu,H}(x)$ with the operator $\operatorname{Id} - (J\frac{d}{dt} + \mu)^{-1}(\theta \operatorname{Id} + \mu), \theta \neq 2\pi\mathbb{Z}$ we prove that we can relate $\deg_{LS}(\Phi_{\mu,H}, \Omega, 0)$ with the braid Floer homology: a delicate analysis of $\deg_{LS}(\Phi_{\mu,H}, \Omega, 0)$ via spectral flow theory reveals that

$$\deg_{LS}(\Phi_{\mu,H}, \Omega, 0) = -\chi(\mathrm{HF}_*([x]_{\mathbb{D}^2} \operatorname{rel} y)) = -\chi(\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2})), \tag{1.7}$$

where χ is the Euler characteristic of the braid Floer homology. In (1.7) the second equality follows from invariance of fibers of the braid Floer homology. The parallel with the finite dimensional case is clear. In case of finite dimensions, via the Morse inequalities one defines the Euler-Morse characteristic for *gradient* vector fields and extends it via the Brouwer degree to *arbitrary* vector fields. In our case we give meaning of the Euler-Floer characteristic, naturally associated to the *variational* problem (1.5) to *non-variational* systems such as (1.6), via infinite dimensional degree theory.

The above arguments lead to the definition of an *index* ι for non-degenerate and isolated one-periodic closed integral curves x of X. By using the theory of parity of index zero Fredholm operators we prove that for a non-degenerate and isolated one-periodic closed integral curves of X we have that $\deg_{LS}(\Phi_\mu, \Omega, 0)$ is independent of the choice of μ and of θ. More generally the index $\iota(x)$ is independent of the inversion of the operator $J\frac{d}{dt} + \mu$, and of the choice of any gauging matrix $\Theta \in \mathrm{M}_{2\times 2}(\mathbb{R})$, provided that $\sigma(\Theta) \cap 2\pi i\mathbb{Z} = \varnothing$. We then provide a derivation of a Poincaré-Hopf formula for relative braid classes. The latter has the form (recall (1.1))

$$\sum_{x_0} \iota(x_0) = \chi(\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2})).$$

The sum here is computed over all closed integral curves $x_0 \operatorname{rel} y$ in the proper relative braid class fiber $[x]_{\mathbb{D}^2} \operatorname{rel} y$. The index formula can be used to obtain existence results for closed integral curves of arbitrary vector fields in proper relative braid classes and provides an extension of the already mentioned forcing result contained in [50]: if $\chi(\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2})) \neq 0$, this forces the existence of closed integral curves of arbitrary vector fields X in any proper relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$. In the language of diffeomorphisms and periodic points the result can be reformulated as follows: under the hypotheses that a diffeomorphism g has an invariant set A_m representing the m-strand braid class $[y]_{\mathbb{D}^2}$, for any proper relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$ for which the Euler characteristic of the braid Floer homology does not vanish, there exists a fixed point for g such that the union $A_m \cup \{x\}$ represents the relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$. Note that we obtain results concerning fixed points of diffeomorphisms, but the same theory, with small but necessary adjustments, applies to periodic points of the diffeomorphisms. A further development would be to extend the result to any two-dimensional surfaces (with or without boundary).

The remaining part of Chapter 2 deals with computability of the Euler-Floer characteristic. The latter can indeed be determined via a discrete topological invariant. In this sense the challenge of constructing an isomorphism which links the Floer homology for proper relative braid classes to the Conley homology of proper discretized relative braid classes via Morse homology is not that far from being solved. On the level of the Euler characteristic of the three homology theories, the following holds

$$\chi\left(\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2})\right) = \chi\left(\mathrm{HM}_*([q \operatorname{rel} Q])\right) = \chi\left(\mathrm{HC}_*([q_D \operatorname{rel} Q_D])\right).$$

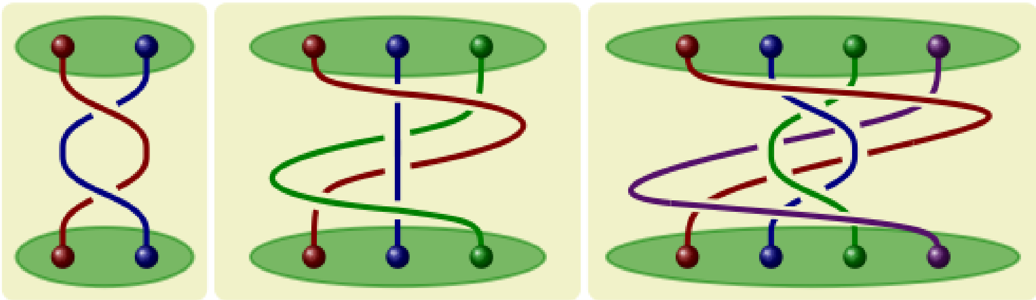

Figure 1.6: Representation of full twists of braids with 2, 3 and 4 strands (source [30]).

The idea behind the proof of this result is to first relate $\chi(\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2}))$ to mechanical Lagrangian systems and then use a discretization approach based on the method of broken geodesics. This result opens the door for computation of the Floer Homology (at least on the level of the Euler characteristic), since the problem of computing $\mathrm{HC}_*([q_D \operatorname{rel} Q_D])$ is combinatorial, and relates the Floer homology to finitely computable simplicial homology.

1.5.2 Braid Floer homology equals braid Morse homology

Chapter 3 consists of an isomorphism theorem between Floer homology for PRBC and Morse homology for Legendrian PRBC. Let $x = (p, q) \in \mathbb{D}^2$ and $y = (P, Q) \in \mathbb{D}^2$, such that $x \operatorname{rel} y$ is a proper relative braid. Compose $x \operatorname{rel} y$ with an integer ℓ of full twists Δ^2. A full twist can be explained informally in the following way: think of pieces of string attached to the tips of your fingers and rotate one hand by π; this is the half-twist, also also called the Garside element. Rotating the hand once more gives the full twist (see Figure 1.6).

If the number ℓ is chosen properly then $(x \operatorname{rel} y) \cdot \Delta^{2\ell}$ gives rise to a braid $x^+ \operatorname{rel} y^+$ with only positive crossings. The latter are called *positive* (relative) braids. This form for $(x \operatorname{rel} y) \cdot \Delta^{2\ell} = x^+ \operatorname{rel} y^+$ is called the Garside normal form, see [10] or [25].

Passing to braid classes, we obtain the following equality

$$[(x \operatorname{rel} y) \cdot \Delta^{2\ell}]_{\mathbb{D}^2} = [x^+ \operatorname{rel} y^+]_{\mathbb{D}^2} \tag{1.8}$$

By the shift property proved in [50] (Property (iii) of Section 1.4.1), on the level of the homology, this yields

$$\mathrm{HF}_{*-2\ell}([(x \operatorname{rel} y) \cdot \Delta^{2\ell}]_{\mathbb{D}^2}) \cong \mathrm{HF}_*([x^+ \operatorname{rel} y^+]_{\mathbb{D}^2}). \tag{1.9}$$

It follows from (1.9) that we can restrict ourselves to positive braids, hence from now on we will consider, without loss of generality, only positive relative braid classes. Positive braids enjoy, up to isotopy, the Legendrian property, in other words $x^+ \operatorname{rel} y^+$ is isotopic to a Legendrian relative braid $x^L \operatorname{rel} y^L$. The latter can be written as $x^L = (q_t, q)$ and $y^L = (Q_t, Q)$. Denoting by π_2 is the projection onto the second coordinate we can write (relative) Legendrian as $q \operatorname{rel} Q$. We denote by $[q \operatorname{rel} Q]$, all the (relative) braids which can be homotoped via a Legendrian isotopy to $q \operatorname{rel} Q$, and by $[q] \operatorname{rel} Q$ the associated fiber.

Having introduced the concepts used in the third chapter, in the following we summarize the content of Chapter 3, which consists of three different sections.

In the first section we define the braid Floer homology with respect to a new class of Hamiltonian functions. The construction is carried out by taking into account a broader class of braid classes. We consider relative classes that are homotopic to $x \operatorname{rel} y$ via homotopies in $\mathbb{R}^2$ instead of $\mathbb{D}^2$, and we denote them by $[x \operatorname{rel} y]_{\mathbb{R}^2}$. In this case the problem comes from the fact that fibers $[x]_{\mathbb{R}^2} \operatorname{rel} y$ are not a-priori bounded, since they are not a priori contained in compact subsets of $\mathbb{D}^2$ as it happens for $[x]_{\mathbb{D}^2} \operatorname{rel} y$. To overcome the issue of non-compactness of $\mathbb{R}^2$, we consider a new class of Hamiltonian functions which we call hyperbolic. Following the construction summarized in Section 1.4.1 we obtain the definition of the *hyperbolic* Floer homology for *unbounded* proper relative braid class, which is denoted by

$$\mathrm{HHF}_*([x \operatorname{rel} y]_{\mathbb{R}^2}).$$

Even though we restrict our attention to positive braids, the hyperbolic braid Floer homology can be defined for all kind of braids, not only for positive ones. By following the arguments in [50] also in this case the shift theorem holds, i.e.

$$\mathrm{HHF}_{*-2\ell}([(x \operatorname{rel} y) \cdot \Delta^{2\ell}]_{\mathbb{R}^2}) \cong \mathrm{HHF}_*([x^+ \operatorname{rel} y^+]_{\mathbb{R}^2}). \tag{1.10}$$

The main result contained in the first section of Chapter 3 consists of proving that

$$\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2}) \cong \mathrm{HHF}_*([x \operatorname{rel} y]_{\mathbb{R}^2}). \tag{1.11}$$

The second part of Chapter 3 deals with Morse homology for braids. This is also a new result: so far, the formulation of a Morse theory for braids has been proven for piecewise linear braids in [28], not yet for continuous ones. By selecting a positive representative $x^+ \operatorname{rel} y^+$ in $[x \operatorname{rel} y]_{\mathbb{R}^2}$, and considering Legendrian isotopies, in the the second part of Chapter 3 we focus our attention on Legendrian relative braid classes $[q \operatorname{rel} Q]_{\mathbb{R}}$. For such braids we construct a Morse-type homology. The fact that, to build our theory, we can use the classical Morse index, instead of the Conley-Zehnder index, derives from the special properties of the Legendrian braid classes, where only positive crossings are admitted. As in the previous case we consider unbounded classes and a special class of Hamiltonians, which, in this case, are called mechanical. The latter allows to construct braid invariants with support on non-compact manifolds. At the end of the second part of Chapter 3, we define

$$\mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}}),$$

i.e., the mechanical Morse homology for unbounded proper Legendrian braid classes. We observe, furthermore, that

$$\mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}}) \cong \mathrm{HM}_*([q \operatorname{rel} Q]), \tag{1.12}$$

where the latter is the Morse analogue of $\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2})$.

In the last part of the chapter we prove that, for a (positive) proper relative braid class $[x \operatorname{rel} y]_{\mathbb{D}^2}$, the following holds:

$$\mathrm{HHF}_*([x \operatorname{rel} y]_{\mathbb{R}^2}) \cong \mathrm{HHM}_*([q \operatorname{rel} Q]_{\mathbb{R}}). \tag{1.13}$$

The isomorphism (1.13) is proved using the machinery of [47], with some modifications that make it applicable to the theory of relative braid classes. In essence, to prove (1.13) we use a perturbation method, through which the solutions of the heat equation can be seen as limit as ε goes to zero of an ε dependent Cauchy-Riemann equation, see Section 3.4.1. As in [47] we prove that the bounded solutions of the Cauchy-Riemann equations are in one-to-one correspondence with the bounded solutions of the heat flow. The map which ensures the one-to-one correspondence takes the name of the Salamon-Weber map. We prove that this map respects the braid classes. As a consequence, the Morse complex defined for Legendrian braid classes agrees, up to isomorphisms, with the Floer complex defined for relative braid classes.

Putting together (1.11) (1.12) and (1.13) we obtain that for a proper positive braid class in $\mathbb{D}^2$ $[x \operatorname{rel} y]_{\mathbb{D}^2}$ it holds that

$$\mathrm{HF}_*([x \operatorname{rel} y]_{\mathbb{D}^2}) \cong \mathrm{HM}_*([q \operatorname{rel} Q]).$$

By considering not-only positive proper relative braid classes, and considering the shift (1.9), this is a first step towards the conjecture that

$$\mathrm{HF}_{*-2\ell}([x \operatorname{rel} y]_{\mathbb{D}^2}) \cong \mathrm{HM}_*([q \operatorname{rel} Q]) \cong \mathrm{HC}_*([q_D \operatorname{rel} Q_D]). \tag{1.14}$$

Equation (1.14) would link the Floer braid invariants to the discrete invariant for piecewise linear positive braid classes, and hence to finitely computable simplicial homology, opening finally the door for computation of the Floer homology.

1.5.3 Asymptotic behavior of the Cauchy-Riemann equations

Chapter 4 goes towards the direction of constructing a Floer homology theory in a non-variational setting. Our result is purely topological and exploits the structure of the Cauchy-Riemann equations. By looking at the construction of the Floer/Morse/Conley homology we see that the Cauchy-Riemann equations are obtained as formal L^2-gradient flow of the Hamiltonian action. In this case, bounded solutions will be, generically, connecting orbits between equilibria. As already mentioned, equilibria (i.e. the critical points of the action functional) are in this case periodic solutions of the equation

$$x_t = X_H(t, x), \quad x \in \mathbb{D}^2, t \in S^1, \tag{1.15}$$

for a chosen non-autonomous Hamiltonian vector field X_H on $\mathbb{D}^2$. For a *general* vector fields X we have shown furthermore that we can build a Poincaré-Hopf formula and give meaning to the Euler-Floer characteristic. By substituting a non-Hamiltonian vector field X in (1.15) we lose the variational structure, and, with it, the gradient-like behavior of the Cauchy-Riemann equations. In this case they become

$$u_s - J(u_t - X(t, u)) = 0, \quad u : \mathbb{R} \times S^1 \to \mathbb{D}^2, t \in S^1. \tag{1.16}$$

If $X = X_H$, then generically bounded solutions of the non-linear CRE are connecting orbits between one-periodic solutions of (1.15). If X is arbitrary, as in (1.16), then a priori bounded solutions do not have the connecting orbit structure, since (1.16) is not a gradient flow. Nevertheless, we have a result concerning the asymptotic of bounded solutions of (1.16). We prove that the asymptotics of (1.16) behaves surprisingly well as time s goes to infinity. More precisely, we prove that bounded solutions of Equation (1.16) admit Poincaré-Bendixson behavior.

The classical Poincaré-Bendixson Theorem describes the asymptotic behavior of flows in the plane. The topology of the plane puts severe restrictions on the behaviour of limit sets. Poincaré-Bendixson Theorem states for example that if the α- and the ω-limit set of a bounded trajectory of a smooth flow in $\mathbb{R}^2$ does not contain equilibria, then the limit set is a periodic orbit. In full generality the classical Poincaré-Bendixson Theorem can be formulated as follows.

1.5.1. Theorem (Poincaré-Bendixson (1906))**.** *Let R be a region of the plane which is closed and bounded. Consider a dynamical system $\dot{x} = X(x)$ in R where the vector field X is at least C^1. Assume that R contains no fixed points of X. Assume furthermore that there exists a trajectory γ of X (a solution of $\dot{x} = X(x)$) starting in R which stays in R for all future times. Then,*

(i) either γ is a closed orbit

(ii) or γ asymptotically approaches a closed orbit.

The classical proof of the Poincaré-Bendixson Theorem exploits the fact that, since the vector field X is autonomous, flow-lines can not intersect. As a consequence, the Jordan curve theorem is applicable and hence restricts the asymptotic behavior of flow-lines in two-dimensional domains. We stress that the above result is strictly linked to the dimensionality of the plane and essentially rules out chaos in the plane. However, it does not seem to hold for other configuration spaces or other types of dynamical systems.

Dynamical systems on two-dimensional manifolds other than the plane may well violate the Poincaré-Bendixson Theorem. Consider for instance the following vector field on the torus, which we identify with the unit square in the plane with opposite sides identified:

$$\dot{x} = 1 \ \text{ and } \ \dot{y} = \pi. \tag{1.17}$$

There is nothing special about the choice of π: any other irrational number would work just as well. Even though the torus is compact and the vector field (1.17) does not have any zeros, the orbits of (1.17) are not periodic: one can check that these orbits densely fill up the torus. This is referred to as quasi-periodic motion. Nevertheless, there is a generalization of the Poincaré-Bendixson for two-dimensional manifolds: either the classical dichotomy holds or the manifold is a torus.

In dimension three or higher, orbits may approach a very complicated limit set known as a *strange attractor*, which is characterized by a non-integer dimension and the fact that the dynamics on it are sensitive to initial conditions. In other words, chaos occurs. A celebrated example of a strange attractor is the *Lorentz attractor*.

However, the remarkable result by Fiedler and Mallet-Paret [18] establishes an extension of the Poincaré-Bendixson Theorem to infinite dimensional dynamical systems with a discrete positive Lyapunov function. They apply their result to scalar parabolic equations of the form

$$u_s = u_{tt} + f(x, u, u_t), \quad x \in S^1, f \in C^2. \tag{1.18}$$

For this equation the result of Matano ([36]) holds: it states that intersection between two solutions of (1.18) can only be destroyed (and not created) as times s increases. Here, the existence of a linear projection onto $\mathbb{R}^2$, of a discrete *positive* Lyapunov function combined with regularity of Equation (1.18) force solutions of (1.18) to have a Poincaré-Bendixson like behavior. As a matter of fact, the result contained in [18] does not only hold for the Equation (1.18), but for regular (semi) flows on Banach spaces endowed with a positive discrete Lyapunov function and a linear projection onto $\mathbb{R}^2$. The result is independent of the dimensionality of the system.

With our result, we establish a version of the Poincaré-Bendixson Theorem for bounded orbits of the non-linear Cauchy-Riemann equations in the plane. We prove that the asymptotic behavior, as s goes to infinity, of bounded solutions of Equation (1.16) is as simple as the limiting behavior of flows in $\mathbb{R}^2$. The non-linear Cauchy Riemann system is elliptic, and the Cauchy problem for elliptic equations is unstable with regard to small variations of data, i.e., it is ill-posed. As a consequence, there is no flow associated to (1.16). For this reason we consider the space of bounded flow-lines of (1.16). This space has nice properties, among which compactness and Hausdorffness. Since Equation (1.16) is autonomous in s, we have that the space of bounded solution is invariant under s-translation. By translating flow-lines we can build a flow on such a space. The constructed flow is not regularizing, but at least it maintain the requirement of continuity. Furthermore, flow lines of equation (1.16) are endowed with a discrete Lyapunov function: as explained in the previous sections as times s increases, the winding number between two solutions decreases, possibly reaching negative values.

By embedding equation (1.16) in a more abstract setting, which include also equation (1.18), our result gives an abstract extension of the Poincaré-Bendixson Theorem to flows that allow a discrete Lyapunov function. We point out that the main differences between the results in [18] for parabolic equations and the results in Chapter 4, are that the Cauchy-Riemann equations do not define a well-posed initial value problem and, more importantly, the discrete Lyapunov functions that we consider are not bounded from below. Furthermore, our result does not assume differentiability of the flow, nor does the flow need to be defined on a compact Banach space. We only assume the space to be compact and Hausdorff. We also believe that most of the result contained in Chapter 4 can be extended to semi-flows.

Our result could be used to build a non-variational Floer theory. By proving that the asymptotic behavior of the non-linear Cauchy-Riemann equations is either a point or a periodic orbit we could build a Floer theory "à la Smale"[49] by incorporating in the chain complex periodic orbits and fixed points.

1.6 Conclusions and future work

The list of challenges we would like to solve is far from being complete. First of all, the question of transversality for a complete Morse theory for Legendrian PRBC has not been proved in the present work. We expect this to hold via modifying the proof for the Hamiltonian case and exploiting the Sard-Smale theory together with a version of the Implicit Function Theorem in infinite dimension. Second, we would like to fully prove the isomorphism (1.14). The first half is contained in this thesis, but the second half has a special meaning: it would open the door for computation of the Floer homology, via the construction of a finite cube complex. Developing computer algorithms would be a further step.

Furthermore, extending the Floer theory to non-variational problems would be even more challenging. In fact, Floer theory has not been applied beyond the variational context, since it crucially uses the gradient structure of the Cauchy-Riemann equations. The above described Monotonicity Principle of the Cauchy-Riemann equations with respect to the crossing number $\mathrm{Cross}([x]_{\mathbb{D}^2}\,\mathrm{rel}\,y)$ remains valid for the non-variational Cauchy-Riemann equation (1.16). As in the variational case, bounded solution of the Cauchy-Riemann equation in a proper relative braid class are isolated. Notwithstanding, in order to link the Floer invariants $\mathrm{HF}_*([x]\,\mathrm{rel}\,y)$ to the non-variational Cauchy-Riemann equations we need to build a complex in a different way. In this sense the Poincaré-Bendixson Theorem for the non-linear Cauchy-Riemann equations suggests that limits as $s \to \infty$ are either periodic solutions of $x_t = X_H(t,x)$ or periodic solutions in s (and t). This first step establishes that the non-variational Cauchy-Riemann equations are generically a Morse-Smale system.

The next step would be, after putting the system in general position, to build an appropriate chain complex (C_*, ∂_*) incorporating periodic orbits and fixed points. If such an extension of the Floer homology can be developed then

$$H_*(C_*, \partial_*) \cong \mathrm{HF}_*([x\,\mathrm{rel}\,y]_{\mathbb{D}^2}).$$